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Completeness of the Dirac oscillator eigenfunctions

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Abstract

Completeness of the Dirac oscillator eigenfunctions is proved in one and three spatial dimensions. Proofs are based on standard properties of the Hermite and the generalized Laguerre polynomials.

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1. Introduction

In recent years various mathematical properties of solutions to the Dirac oscillator eigenproblem have been extensively investigated both analytically and algebraically (e.g., [1–5]). The purpose of the present work is to prove completeness of the Dirac oscillator eigenfunctions. To the best of our knowledge, this problem, very important from the point of view of past [6–8] and planned applications, has not been studied yet.

2. The Dirac oscillator in one spatial dimension

2.1. Eigenproblem and its solutions

The differential eigenproblem for a one-dimensional Dirac oscillator is

$$c\alpha\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}-\mathrm{i}\beta m\omega x\right)\Psi(x)+\beta mc^2\Psi(x)=E\Psi(x)\qquad(-\infty<x<\infty)\tag{2.1}$$

$$\Psi(x)\ \text{bounded for}\ x\to\pm\infty\tag{2.2}$$

where $\omega > 0$ (oscillator frequency) is a fixed parameter, E is an eigenvalue while α and β are 2×2 matrices obeying

$$\alpha^2=\beta^2=I_2\qquad \alpha\beta+\beta\alpha=0\tag{2.3}$$

with I_2 denoting the unit 2×2 matrix. For the present purposes the most convenient representations of α and β are

$$\alpha=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\qquad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\tag{2.4}$$

and expressing the wave function $\Psi(x)$ in terms of its components

$$\Psi(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{2.5}$$

we rewrite the eigensystem (2.1) and (2.2) in the explicit form

$$c\hbar \frac{\mathrm{d}f(x)}{\mathrm{d}x} + mc\omega x f(x) = (mc^2 + E)g(x) \tag{2.6}$$

$$c\hbar \frac{\mathrm{d}g(x)}{\mathrm{d}x} - mc\omega x g(x) = (mc^2 - E)f(x) \tag{2.7}$$

$$f(x) \text{ and } g(x) \text{ bounded for } x \to \pm\infty. \tag{2.8}$$

Solving the eigensystem (2.6)–(2.8) one finds [6] that its solutions may be labeled by a quantum number n assuming *all* integer values. The eigenvalues are

$$E_n = \pm mc^2 \sqrt{1 + 2|n| \frac{\hbar\omega}{mc^2}} \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{2.9}$$

where the upper sign should be chosen for $n \geqslant 0$ and the lower one for $n < 0$, while components of the eigenfunctions, orthonormal in the sense of

$$\int_{-\infty}^{\infty} \mathrm{d}x \ [f_n(x) f_m(x) + g_n(x) g_m(x)] = \delta_{nm} \tag{2.10}$$

are

$$f_n(x) = \sqrt{\frac{\lambda(E_n + mc^2)}{2^{|n|+1}|n|!\sqrt{\pi}E_n}}\, H_{|n|}(\lambda x)\mathrm{e}^{-\lambda^2 x^2/2} \tag{2.11}$$

$$g_n(x) = \pm\sqrt{\frac{\lambda(E_n - mc^2)}{2^{|n|}(|n|-1)!\sqrt{\pi}E_n}}\, H_{|n|-1}(\lambda x)\mathrm{e}^{-\lambda^2 x^2/2} \tag{2.12}$$

(observe that $g_0(x) \equiv 0$) where $H_k(\xi)$ is the Hermite polynomial [9] and

$$\lambda = \sqrt{\frac{m\omega}{\hbar}}. \tag{2.13}$$

It will be profitable to notice the following symmetry properties of the eigensolutions:

$$E_{-n} = -E_n \qquad (n \neq 0) \tag{2.14}$$

$$f_{-n}(x) = \sqrt{\frac{E_n - mc^2}{E_n + mc^2}}\, f_n(x) \qquad g_{-n}(x) = -\sqrt{\frac{E_n + mc^2}{E_n - mc^2}}\, g_n(x) \qquad (n \neq 0). \tag{2.15}$$

2.2. *Proof of completeness*

Completeness of the one-dimensional Dirac eigenfunctions will be established if we succeed in proving the closure relation

$$\sum_{n=-\infty}^{\infty} \begin{pmatrix} f_n(x) \\ g_n(x) \end{pmatrix} \begin{pmatrix} f_n(x') & g_n(x') \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - x') \qquad (-\infty < x, x' < \infty) \tag{2.16}$$

where $\delta(x - x')$ is the Dirac delta function. To this end, we observe that the matrix relation (2.16) is equivalent to two 'diagonal' scalar relations

$$I(x, x') \equiv \sum_{n=-\infty}^{\infty} f_n(x) f_n(x') = \delta(x - x') \tag{2.17}$$

$$J(x, x') \equiv \sum_{n=-\infty}^{\infty} g_n(x) g_n(x') = \delta(x - x') \tag{2.18}$$

and one 'off-diagonal' scalar relation

$$K(x, x') \equiv \sum_{n=-\infty}^{\infty} f_n(x) g_n(x') = 0 \tag{2.19}$$

(the second 'off-diagonal' relation follows immediately from the relation (2.19)).

To prove the relation (2.17), we decompose its left-hand side in the following way:

$$I(x, x') = f_0(x) f_0(x') + \sum_{n=1}^{\infty} \left[f_n(x) f_n(x') + f_{-n}(x) f_{-n}(x') \right]. \tag{2.20}$$

On substituting here the explicit forms (2.11) of f_n and collecting terms containing the Hermite polynomials of the same degree, we arrive at

$$I(x, x') = \sum_{n=0}^{\infty} \frac{\lambda}{2^n n! \sqrt{\pi}} \, \mathrm{e}^{-\lambda^2 (x^2 + x'^2)/2} H_n(\lambda x) H_n(\lambda x'). \tag{2.21}$$

Application of the following known closure relation obeyed by the normalized Hermite functions

$$\sum_{n=0}^{\infty} \frac{1}{2^n n! \sqrt{\pi}} \, \mathrm{e}^{-(\xi^2 + \xi'^2)/2} H_n(\xi) H_n(\xi') = \delta(\xi - \xi') \qquad (-\infty < \xi, \xi' < \infty) \tag{2.22}$$

and the following identity obeyed by the Dirac delta function

$$\delta(\lambda x - \lambda x') = \lambda^{-1} \delta(x - x') \qquad (-\infty < x, x' < \infty) \tag{2.23}$$

to the right-hand side of equation (2.21) immediately leads to equation (2.17).

To prove the relation (2.18), we make use of the fact that $g_0(x) \equiv 0$ and decompose the left-hand side of (2.18) as follows:

$$J(x, x') = \sum_{n=0}^{\infty} \left[g_{n+1}(x) g_{n+1}(x') + g_{-n-1}(x) g_{-n-1}(x') \right]. \tag{2.24}$$

Substituting here the explicit forms (2.12) of g_n and collecting terms containing the Hermite polynomials of the same degree, we find

$$J(x, x') = \sum_{n=0}^{\infty} \frac{\lambda}{2^n n! \sqrt{\pi}} \, \mathrm{e}^{-\lambda^2 (x^2 + x'^2)/2} H_n(\lambda x) H_n(\lambda x') \tag{2.25}$$

hence (cf. equations (2.22) and (2.23)) equation (2.18) follows immediately.

Finally, to prove the relation (2.19), we again make use of the fact that $g_0(x) \equiv 0$ and write

$$K(x, x') = \sum_{n=1}^{\infty} \left[f_n(x) g_n(x') + f_{-n}(x) g_{-n}(x') \right]. \tag{2.26}$$

Since equation (2.15) implies that the product $f_n(x) g_n(x')$ is an *odd* function of n, the summand in equation (2.26) vanishes and thus equation (2.19) has been proved.

3. The Dirac oscillator in three spatial dimensions

3.1. Eigenproblem and its solutions

Now we turn to the three-dimensional Dirac oscillator. The relevant eigenproblem is

$$c\boldsymbol{\alpha}\cdot[-\mathrm{i}\hbar\boldsymbol{\nabla}-\mathrm{i}\beta m\omega\boldsymbol{r}]\Psi(\boldsymbol{r})+\beta mc^2\Psi(\boldsymbol{r})=E\Psi(\boldsymbol{r}) \qquad (\boldsymbol{r}\in\mathbb{R}^3) \tag{3.1}$$

$$\Psi(\boldsymbol{r}) \text{ bounded everywhere} \tag{3.2}$$

where again $\omega > 0$ (the frequency of the oscillator) is a fixed parameter and E is an eigenvalue. In equation (3.1) $\boldsymbol{\alpha}$ and β are 4×4 Dirac matrices [10].

The eigensystem (3.1) and (3.2) possesses solutions of the form

$$\Psi_{\kappa m_j}(\boldsymbol{r})=\frac{1}{r}\begin{pmatrix} P_\kappa(r)\,\Omega_{\kappa m_j}(\boldsymbol{n}_r) \\ \mathrm{i}Q_\kappa(r)\,\Omega_{-\kappa m_j}(\boldsymbol{n}_r)\end{pmatrix} \tag{3.3}$$

where $\boldsymbol{n}_r=\boldsymbol{r}/r$, $\Omega_{\pm\kappa m_j}(\boldsymbol{n}_r)$ are spherical spinors, $\kappa=\pm1,\pm2,\ldots$ and $m_j=-|\kappa|+\frac{1}{2},-|\kappa|+\frac{3}{2},\ldots,|\kappa|-\frac{1}{2}$. The radial functions $P_\kappa(r)$ and $Q_\kappa(r)$ are solutions of the eigensystem

$$c\hbar\frac{\mathrm{d}P_\kappa(r)}{\mathrm{d}r}+c\hbar\frac{\kappa}{r}P_\kappa(r)+mc\omega rP_\kappa(r)=(mc^2+E_\kappa)Q_\kappa(r) \qquad (0<r<\infty) \tag{3.4}$$

$$c\hbar\frac{\mathrm{d}Q_\kappa(r)}{\mathrm{d}r}-c\hbar\frac{\kappa}{r}Q_\kappa(r)-mc\omega rQ_\kappa(r)=(mc^2-E_\kappa)P_\kappa(r) \qquad (0<r<\infty) \tag{3.5}$$

$$P_\kappa(0)=Q_\kappa(0)=0 \qquad P_\kappa(r) \text{ and } Q_\kappa(r) \text{ bounded for } r\to\infty. \tag{3.6}$$

The structure of the spectrum of the radial eigensystem (3.4)–(3.6) depends on the sign of κ. The eigenvalues are found [1, 3] to be

$$E_{n\kappa}=\pm mc^2\sqrt{1+4|n|\frac{\hbar\omega}{mc^2}} \qquad (n=0,\pm1,\pm2,\ldots) \qquad \text{for } \kappa<0 \tag{3.7}$$

with the upper sign chosen for $n\geqslant 0$ and the lower one for $n<0$, and

$$E_{n\kappa}=\pm mc^2\sqrt{1+4(|n|+l+\tfrac{1}{2})\frac{\hbar\omega}{mc^2}} \qquad (n=\pm0,\pm1,\pm2,\ldots) \qquad \text{for } \kappa>0 \tag{3.8}$$

with the upper sign chosen for $n=+0,+1,+2,\ldots$ and the lower one for $n=-0,-1,-2,\ldots$. (Notice that for $\kappa>0$ it is necessary to distinguish between the cases $n=+0$ and $n=-0$.) Associated radial eigenfunctions, normalized according to

$$\int_0^\infty \mathrm{d}r\ [P_{n\kappa}(r)P_{m\kappa}(r)+Q_{n\kappa}(r)Q_{m\kappa}(r)]=\delta_{nm} \tag{3.9}$$

are [3]

$$P_{n\kappa}(r)=\sqrt{\frac{\lambda|n|!(E_{n\kappa}+mc^2)}{\Gamma(|n|+l+\frac{3}{2})E_{n\kappa}}}\,(\lambda r)^{l+1}\mathrm{e}^{-\lambda^2r^2/2}L_{|n|}^{(l+1/2)}(\lambda^2r^2) \tag{3.10}$$

$$Q_{n\kappa}(r)=\pm\mathrm{sgn}(\kappa)\sqrt{\frac{\lambda|n'|!(E_{n\kappa}-mc^2)}{\Gamma(|n'|+l'+\frac{3}{2})E_{n\kappa}}}\,(\lambda r)^{l'+1}\mathrm{e}^{-\lambda^2r^2/2}L_{|n'|}^{(l'+1/2)}(\lambda^2r^2) \tag{3.11}$$

where $L_k^{(\alpha)}(\rho)$ is a generalized Laguerre polynomial [9]. (The same sign convention as in equations (3.7) and (3.8) applies in equation (3.11).) It is to be observed that for $\kappa<0$ one has $Q_{0\kappa}(r)\equiv 0$. The integers $|n'|$, l and l' appearing in equations (3.7), (3.8), (3.10) and (3.11) are defined by

$$|n'|=\begin{cases}|n|-1 & \text{for } \kappa<0\\ |n| & \text{for } \kappa>0\end{cases} \tag{3.12}$$

$$l = |\kappa + \tfrac{1}{2}| - \tfrac{1}{2} = \begin{cases} -\kappa - 1 & \text{for } \kappa < 0 \\ \kappa & \text{for } \kappa > 0 \end{cases} \tag{3.13}$$

$$l' = |\kappa - \tfrac{1}{2}| - \tfrac{1}{2} = \begin{cases} -\kappa = l + 1 & \text{for } \kappa < 0 \\ \kappa - 1 = l - 1 & \text{for } \kappa > 0. \end{cases} \tag{3.14}$$

In the next subsection we shall make use of the following symmetry relations:

$$E_{-n\kappa} = -E_{n\kappa} \qquad (\text{except for } n = 0 \text{ when } \kappa < 0) \tag{3.15}$$

$$P_{-n\kappa}(r) = \sqrt{\frac{E_{n\kappa} - mc^2}{E_{n\kappa} + mc^2}}\, P_{n\kappa}(r) \qquad Q_{-n\kappa}(r) = -\sqrt{\frac{E_{n\kappa} + mc^2}{E_{n\kappa} - mc^2}}\, Q_{n\kappa}(r)$$
$$(\text{except for } n = 0 \text{ when } \kappa < 0) \tag{3.16}$$

stemming from equations (3.7), (3.8), (3.10) and (3.11).

3.2. *Proof of completeness*

The supposed closure relation obeyed by radial eigenfunctions is

$$\sum_{n=-\infty}^{\infty} \begin{pmatrix} P_{n\kappa}(r) \\ Q_{n\kappa}(r) \end{pmatrix} \begin{pmatrix} P_{n\kappa}(r') & Q_{n\kappa}(r') \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(r - r') \qquad (0 < r, r' < \infty). \tag{3.17}$$

We shall prove it by deriving the following summation formulae:

$$I_\kappa(r, r') \equiv \sum_{n=-\infty}^{\infty} P_{n\kappa}(r) P_{n\kappa}(r') = \delta(r - r') \tag{3.18}$$

$$J_\kappa(r, r') \equiv \sum_{n=-\infty}^{\infty} Q_{n\kappa}(r) Q_{n\kappa}(r') = \delta(r - r') \tag{3.19}$$

$$K_\kappa(r, r') \equiv \sum_{n=-\infty}^{\infty} P_{n\kappa}(r) Q_{n\kappa}(r') = 0. \tag{3.20}$$

It is to be remembered that in equations (3.17)–(3.20), as well as in the rest of this subsection, the summations over n, extending from $-\infty$ to $+\infty$, for $\kappa < 0$ include a term corresponding to $n = 0$ while for $\kappa > 0$ include terms corresponding to $n = +0$ and $n = -0$.

We begin with the relation (3.18). If $\kappa < 0$, we decompose the left-hand side of that equation in the following way:

$$I_{-l-1}(r, r') = P_{0,-l-1}(r) P_{0,-l-1}(r') + \sum_{n=1}^{\infty} \left[P_{n,-l-1}(r) P_{n,-l-1}(r') + P_{-n,-l-1}(r) P_{-n,-l-1}(r') \right] \tag{3.21}$$

while if $\kappa > 0$, we write

$$I_l(r, r') = \sum_{n=+0}^{\infty} \left[P_{nl}(r) P_{nl}(r') + P_{-nl}(r) P_{-nl}(r') \right]. \tag{3.22}$$

In either case, collecting terms containing the generalized Laguerre polynomials of the same degree, we arrive at

$$I_\kappa(r, r') = \sum_{n=0}^{\infty} \frac{2\lambda n!}{\Gamma(n + l + \frac{3}{2})} (\lambda r)^{l+1} (\lambda r')^{l+1} \mathrm{e}^{-\lambda^2 (r^2 + r'^2)/2} L_n^{(l+1/2)}(\lambda^2 r^2) L_n^{(l+1/2)}(\lambda^2 r'^2). \tag{3.23}$$

The series in equation (3.23) may be summed on making use of the known closure relation

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} (\rho\rho')^{\alpha/2} \mathrm{e}^{-(\rho+\rho')/2} L_n^{(\alpha)}(\rho) L_n^{(\alpha)}(\rho') = \delta(\rho - \rho') \qquad (0 < \rho, \rho' < \infty) \tag{3.24}$$

obeyed by the generalized Laguerre functions. Subsequent application of the relationship

$$\delta(\lambda^2 r^2 - \lambda^2 r'^2) = \frac{\delta(r - r')}{2\lambda^2 (rr')^{1/2}} \qquad (0 < r, r' < \infty) \tag{3.25}$$

following from the standard properties of the Dirac delta function, completes the proof of equation (3.18).

Next we consider equation (3.19). When $\kappa < 0$ we utilize the fact that $Q_{0,-l-1}(r) \equiv 0$ and rewrite the left-hand side of this equation in the form

$$J_{-l-1}(r, r') = \sum_{n=0}^{\infty} \left[Q_{n+1,-l-1}(r) Q_{n+1,-l-1}(r') + Q_{-n-1,-l-1}(r) Q_{-n-1,-l-1}(r') \right] \tag{3.26}$$

while for $\kappa > 0$ we decompose it in the following way:

$$J_l(r, r') = \sum_{n=+0}^{\infty} \left[Q_{nl}(r) Q_{nl}(r') + Q_{-nl}(r) Q_{-nl}(r') \right]. \tag{3.27}$$

On collecting terms at the generalized Laguerre polynomials of the same degree, in both cases we find

$$J_\kappa(r, r') = \sum_{n=0}^{\infty} \frac{2\lambda n!}{\Gamma(n + l' + \frac{3}{2})} (\lambda r)^{l'+1} (\lambda r')^{l'+1} \mathrm{e}^{-\lambda^2 (r^2 + r'^2)/2} L_n^{(l'+1/2)}(\lambda^2 r^2) L_n^{(l'+1/2)}(\lambda^2 r'^2). \tag{3.28}$$

Hence and from equations (3.24) and (3.25) we obtain equation (3.19).

To prove equation (3.20), for $\kappa < 0$ we rewrite its left-hand side as

$$K_{-l-1}(r, r') = \sum_{n=1}^{\infty} \left[P_{n,-l-1}(r) Q_{n,-l-1}(r') + P_{-n,-l-1}(r) Q_{-n,-l-1}(r') \right] \tag{3.29}$$

and for $\kappa > 0$ as

$$K_l(r, r') = \sum_{n=+0}^{\infty} \left[P_{nl}(r) Q_{nl}(r') + P_{-nl}(r) Q_{-nl}(r') \right]. \tag{3.30}$$

Since equation (3.16) implies that the product $P_{n\kappa}(r) Q_{n\kappa}(r')$ is an odd function of n, the summands in equations (3.29) and (3.30) vanish and we arrive at equation (3.20).

The closure relation (3.17) and the known closure relation

$$\sum_{\substack{\kappa=-\infty \\ (\kappa \neq 0)}}^{\infty} \sum_{m_j=-|\kappa|+1/2}^{|\kappa|-1/2} \Omega_{\kappa m_j}(\boldsymbol{n}_r) \Omega_{\kappa m_j}^{\dagger}(\boldsymbol{n}'_r) = I_2\, \delta^{(2)}(\boldsymbol{n}_r - \boldsymbol{n}'_r) \tag{3.31}$$

obeyed by the spherical spinors, together with the following well known representation of the three-dimensional Dirac delta function

$$\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}') = \frac{\delta(r - r') \delta^{(2)}(\boldsymbol{n}_r - \boldsymbol{n}'_r)}{rr'} \tag{3.32}$$

immediately imply the closure relation for the four-component eigenfunctions (3.3)

$$\sum_{\substack{\kappa=-\infty \\ (\kappa\neq 0)}}^{\infty} \sum_{m_j=-|\kappa|+1/2}^{|\kappa|-1/2} \sum_{n=-\infty}^{\infty} \Psi_{n\kappa m_j}(\boldsymbol{r})\Psi^{\dagger}_{n\kappa m_j}(\boldsymbol{r}') = I_4\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}') \tag{3.33}$$

where I_4 is the unit 4×4 matrix.

We leave to the reader as an exercise to prove completeness of the Dirac oscillator eigenfunctions in two spatial dimensions [4].

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